CENTER FOR POLICY RESEARCH
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WORKING PAPER SERIES

Testing for Spatial Correlation under a Complete Bipartite Network

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Paper No. 264
July 2024

ISSN: 1525-3066

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Abstract

This note shows that for a spatial regression with a weight matrix depicting a complete bipartite network, the Moran *I* test for zero spatial correlation is never rejected when the alternative is positive spatial correlation no matter how large the true value of the spatial correlation coefficient. In contrast, the null hypothesis of zero spatial correlation is always rejected (with probability one asymptotically) when the alternative is negative spatial correlation and the true value of the spatial correlation coefficient is near -1.

JEL No.: C12; C21; C31.

Keywords: Spatial Error Model; Moran I Test, Complete Bipartite Network.

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1 Introduction

Several papers have pointed out that some special spatial weighting matrices cause problems in testing for zero spatial correlation. For example, one popular special spatial weighting matrix is the *equal weight matrix* that has zero elements across the diagonal and equal elements $1/(n-1)$ off the diagonal, where n denotes the sample size. In this scenario, everybody in the sample is everybody's neighbor and affects his or her neighbor equally. Such a weighting matrix was considered by Case (1992), Kelejian and Prucha (2002), Kelejian et al. (2006) and Baltagi (2006), to name a few. For this equal weight matrix, Baltagi and Liu (2009) showed that the Lagrange Multiplier (LM) test for spatial lag dependence is always equal to $n/(2n-1)$ and is not a function of the spatial parameter ρ. This means that this LM test statistic converges to $1/2$ as $n \to \infty$, no matter what the true value of the spatial correlation coefficient ρ is. It also means that zero spatial lag correlation is never rejected for all values of ρ. Martellosio (2011) further showed that any invariant test of equal weights spatial dependence must have power equal to its size.

In this paper, we consider another special spatial weighting matrix that is used in describing the *complete bipartite network*, where individuals in the sample are divided into two blocks numbering p and q with $p+q=n$. In this network, each individual in one block is connected to all individuals in the other block but not connected to any individual in the same block. This weighting matrix has been used by Jackson (2008), Bramoullé, Djebbari and Fortin (2009), Lee, Liu and Lin (2010), Lin (2010), Blume, Brock, Durlauf and Ioannides (2011), Beckett (2016), Hillier and Martellosio (2018), Rödder, Dellnitz, Kulmann, Litzinge and Reucher (2019), Hsieh, Lin and Patacchini (2020), Li, Cao, Li, Tan and Meng (2022) and Martellosio (2022), to name a few. When $p=1$ or $q=1$, it reduces to the *star network*, a particularly important case in network theory, where one individual is connected to all other individuals in a group and all the other individuals in the group connect only to him. For the spatial error model with this *complete bipartite network* weight matrix, we show that asymptotically, the Moran I test, which tests the null of no spatial correlation, can never reject the null hypothesis against positive spatial autocorrelation no matter how large the true value of the spatial correlation coefficient. In contrast, the Moran I test will always reject the null hypothesis of zero spatial correlation against negative spatial autocorrelation, (with probability 1 asymptotically) when the true value of ρ is near -1.

2 Model and Results

Consider the following linear regression with spatially correlated error term:

$$y_n = X_n\beta + u_n \tag{1}$$

and

$$u_n = \rho W_n u_n + \varepsilon_n, \tag{2}$$

where y_n is an $n \times 1$ vector for the dependent variable. ι_n is a vector of ones of dimension n. X_n is an $n \times k$ matrix of exogenous variables including a constant. β is a $k \times 1$ vector of parameters. ρ is a scalar parameter between -1 and 1. u_n and ε_n are $n \times 1$ vectors, where ε_n is independent and identically distributed as Normal with zero mean and variance σ^2. The $n \times n$ spatial weight matrix W_n is row normalized and has zero elements across the diagonal, see Anselin (1988) and Anselin and Bera (1998) for an excellent treatment of this subject. Define $B_n = I_n - \rho W_n$, where I_n is an identity matrix of dimension n. The spatial error term in Equation (2) can be rewritten as $u_n = B_n^{-1}\varepsilon_n$ so that $E\left(u_n u_n'\right) = \sigma^2 \left(B_n' B_n\right)^{-1}$. The Moran I test statistic for the null hypothesis of $H_0 : \rho = 0$ is given by:

$$I = \frac{\hat{u}_n' W_n \hat{u}_n}{\hat{u}_n' \hat{u}_n} \tag{3}$$

where $\hat{u}_n$ is the OLS residual from Equation (1), see Cliff and Ord (1972). This Moran I test has been well studied by Burridge (1980), Anselin (1988), Kelejian and Prucha (2001), Krämer (2005), Martelosio (2010, 2012) and Baltagi and Yang (2013), to mention a few. As shown in Baltagi and Yang (2013), the standerdized Moran I statistic is asymptotically distributed as $N(0,1)$ under the null hypothesis of $H_0 : \rho = 0$. To be specific, let

$$I^* = \frac{I - \mu_I}{\sigma_I}, \tag{4}$$

where $\mu_I = \frac{1}{n-k} tr\left(M_n W_n\right)$ and $\sigma_I = \sqrt{\frac{tr(M_n W_n M_n W_n') + tr(M_n W_n M_n W_n) - \frac{2}{n-k}[tr(M_n W_n)]^2}{(n-k)(n-k+2)}}$, where $M_n = I_n - P_n$ with $P_n = X_n \left(X_n' X_n\right)^{-1} X_n'$. We have $I^* \xrightarrow{d} N(0,1)$ under the null hypothesis.

In this paper, we will consider the spatial weighting matrix that corresponds to the *complete bipartite network*, where individuals in the sample are divided into two blocks such that each individual in one block is connected to all individuals in the other block but to none in the same block. To be specific, the complete bipartite network spatial weighting matrix is as follows:

$$W_n = \begin{bmatrix} 0_p & \frac{1}{q}\iota_p \iota_q' \\ \frac{1}{p}\iota_q \iota_p' & 0_q \end{bmatrix}, \tag{5}$$

where ι_p and ι_q are vectors of ones of dimension p and q, respectively, with $p+q=n$. 0_p and 0_q are matrices of zeros of dimension $p \times p$ and $q \times q$, respectively. This weighting matrix has been studied by Jackson (2008), Bramoullé, Djebbari and Fortin (2009), Lee, Liu and Lin (2010), Lin (2010), Blume, Brock, Durlauf and Ioannides (2011), Beckett (2016), Hillier and Martellosio (2018), Rödder, Dellnitz, Kulmann, Litzinge and Reucher (2019), Hsieh, Lin and Patacchini (2020), Li, Cao, Li, Tan and Meng (2022) and Martellosio (2022), to name a few. When $p=1$ or $q=1$, it reduces to the *star network*, a particularly important case in network theory, where one individual is connected to all other individuals in a group and all the others in the group connect only to him. In what follows, for the complete bipartite network spatial weighting matrix defined in Equation (5), we derive the asymptotic power of the Moran I test statistic against positive or negative spatial autocorrelation, respectively. As shown in Lee, Liu and Lin (2010),

$$W_n^2 = \begin{bmatrix} \frac{1}{p}\iota_p\iota_p' & 0_{pq} \\ 0_{qp} & \frac{1}{q}\iota_q\iota_q' \end{bmatrix} = \begin{bmatrix} \bar{J}_p & 0_{pq} \\ 0_{qp} & \bar{J}_q \end{bmatrix}, \tag{6}$$

where $\bar{J}_p = \frac{1}{p}\iota_p\iota_p'$, $\bar{J}_q = \frac{1}{q}\iota_q\iota_q'$, 0_{pq}, 0_{qp} are matrices of zeros of dimension $p \times q$ and $q \times p$, respectively. Note that

$$W_n + W_n^2 = \begin{bmatrix} \frac{1}{p}\iota_p\iota_p' & \frac{1}{q}\iota_p\iota_q' \\ \frac{1}{p}\iota_q\iota_p' & \frac{1}{q}\iota_q\iota_q' \end{bmatrix} = \begin{bmatrix} \frac{1}{p}\iota_n\iota_p' & \frac{1}{q}\iota_n\iota_q' \end{bmatrix} = \iota_n \begin{bmatrix} \frac{1}{p}\iota_p' & \frac{1}{q}\iota_q' \end{bmatrix}.$$

Hence $M_n\left(W_n + W_n^2\right) = M_n\iota_n \begin{bmatrix} \frac{1}{p}\iota_p' & \frac{1}{q}\iota_q' \end{bmatrix} = 0$ using $M_n\iota_n = 0$ when ι_n is included in X_n so that $M_nW_n = -M_nW_n^2$. Together with the fact $\hat{u}_n = M_nu_n$, we have $\hat{u}_n'W_n\hat{u}_n = u_n'M_nW_nM_nu_n = -u_n'M_nW_n^2M_nu_n = -\hat{u}_n'W_n^2\hat{u}_n$. It is easy to see that W_n^2 is symmetric and idempotent since $\bar{J}_p$ and $\bar{J}_q$ are symmetric and idempotent. Hence $\hat{u}'W^2\hat{u} \geq 0$. Therefore,

$$I = \frac{\hat{u}_n'W_n\hat{u}_n}{\hat{u}_n'\hat{u}_n} = -\frac{\hat{u}_n'W_n^2\hat{u}_n}{\hat{u}_n'\hat{u}_n} \leq 0$$

and hence

$$I^* = \frac{I-\mu_I}{\sigma_I} \leq -\frac{\mu_I}{\sigma_I}.$$

Rewrite $X_n = \begin{bmatrix} X_p \\ X_q \end{bmatrix}$, let $\bar{X}_p = \frac{1}{p}\iota_p'X_p$, $\bar{X}_q = \frac{1}{q}\iota_q'X_q$ and $\bar{X} = \frac{1}{n}\iota_n'X_n$. Define $E_n = I_n - \bar{J}_n$ and $\bar{J}_n = \frac{1}{n}\iota_n\iota_n'$ where ι_n be a vector of ones of dimension n. Let $\tilde{X}_n = E_nX_n = X_n - \iota_n\bar{X}$. We assume the following:

Assumption 1 *We assume* $\left\| \frac{\sqrt{pq}}{n}\left(\bar{X}_p - \bar{X}_q\right) \right\| = o_p(1)$. *In addition,* $plm_{n\to\infty}\frac{1}{n}\tilde{X}_n'\tilde{X}_n$ *exists and is a positive definite matrix.*

Assumption 1 assumes the difference between $\bar{X}_p$ and $\bar{X}_q$ is small. When $X_n = \iota_n$ for example, $\bar{X}_p = \bar{X}_q = 1$ so that $\bar{X}_p - \bar{X}_q = 0$. Also note that $\frac{\sqrt{pq}}{n} \leq \frac{p+q}{2n} = \frac{1}{2}$, where the equality holds when $p = q = \frac{n}{2}$. For a star network for example, where $p = 1$ and $q = n-1$, $\frac{\sqrt{pq}}{n} = \frac{\sqrt{n-1}}{n} \approx \frac{1}{\sqrt{n}}$. To test the null hypothesis of no spatial correlation, i.e., $H_0 : \rho = 0$ against the alternative hypothesis of positive spatial autocorrelation, i.e., $H_1 : \rho > 0$. One rejects H_0 if $I^* > 1.645$. However, in the following theorem, we show that $-\frac{\mu_I}{\sigma_I} = \frac{1}{\sqrt{2}} + o_p(1)$ as $n \to \infty$, the null will never be rejected against the alternative hypothesis of positive spatial correlation, and the test has no power no matter how large ρ is.

Theorem 1 *For the complete bipartite network spatial weighting matrix, under Assumption 1, we have*

$$\lim_{n\to\infty} \Pr\left(I^* > \frac{1}{\sqrt{2}}\right) = 0$$

for all ρ.

The proof is given in the supplemental Appendix available upon request from the authors. Theorem 1 implies that I^* is always bounded by $\frac{1}{\sqrt{2}}$ as $n \to \infty$. Since $1.645 > \frac{1}{\sqrt{2}}$, the null hypothesis is never rejected against the alternative hypothesis of $\rho > 0$, no matter how large ρ is.[1]

To test the null hypothesis of no spatial correlation, i.e., $H_0 : \rho = 0$, against the alternative of negative spatial autocorrelation, i.e., $H_1 : \rho < 0$. One rejects H_0 if $I^* < -1.645$. The following theorem shows that as $n \to \infty$, and ρ is close to -1, the null hypothesis of no spatial correlation is always rejected against the alternative hypothesis of negative spatial correlation and the asymptotic power of the Moran I test is 1.

Theorem 2 *For the complete bipartite network spatial weighting matrix, under Assumption 1, if $\rho = -1 + \frac{1}{\psi_n}$, where $\psi_n \to \infty$ as $n \to \infty$, we have*

$$\lim_{n\to\infty} \Pr\left(I^* < \eta\right) = 1$$

for every constant $\eta < 0$.

The proof is given in the supplemental Appendix available upon request from the authors. For time series, Phillips and Magdalinos (2007) derived the asymptotic theory for the near-unit root case. Lee and Yu (2013) and Baltagi, Kao and Liu (2013) extended the near-unit root case to spatial regression models. In particular, Theorem 3 in Baltagi, Kao and Liu (2013) showed that the QMLE of ρ has a faster convergence rate when the spatial error is near nonstationary. In the proof of Theorem 2, we showed that $I^* \xrightarrow{p} -\infty$ if

[1] It is worth pointing out that Theorem 1 is for the Moran I test on the regression residuals. It may not hold if one uses the Moran I test on the original data. In addition, as we mentioned earlier, the inclusion of the intercept term in X_n is crucial to Theorem 1 as it ensures $M_n W_n = -M_n W_n^2$.

$\psi_n \to \infty$ as $n \to \infty$. Theorem 2 implies that as $n \to \infty$, and ρ is close to -1, the null hypothesis of no spatial correlation is always rejected against the alternative of negative spatial correlation with probability 1 asymptotically. It is worth pointing out that the result in Theorem 2 holds in general, see Theorem 2 in Kelejian and Prucha (2001) which provides a general result for the power of the test to approach 1 as $n \to \infty$. One can verify that the condition in Kelejian and Prucha (2001) is satisfied when $\rho = -1 + \frac{1}{\psi_n}$.

3 Monte Carlo Simulation

Following Baltagi and Yang (2013), we generate the data from Equations (1) and (2), where $X_n = (\iota_n, x_{1n}, x_{2n})$ and $\beta = (5, 1, 1)'$. ι_n is a vector of ones of dimension n. x_{1n}, x_{2n} and ε_n are $n \times 1$ vectors with elements $x_{1i} \overset{iid}{\sim} \sqrt{6}U(0,1)$, $x_{2i} \overset{iid}{\sim} N(0,1)/\sqrt{2}$ and $\varepsilon_i \overset{iid}{\sim} N(0,1)$, respectively. ρ varies over the range $(-0.99, -0.9, -0.6, -0.3, 0, 0.3, 0.6, 0.9, 0.99)$. We let $\frac{p}{n} = 0.3$.[2] The sample sizes considered are $n = (50, 200)$. For each experiment, we perform 10,000 replications. For each data generating process, we report the performance of the standardized Moran I test statistic I^*.

Table 1 reports the summary statistics of I^* and the empirical frequency of $I^* > 1.645$ corresponding to the rejection rates of the null hypothesis $H_0 : \rho = 0$ against the alternative hypothesis of positive spatial autocorrelation, i.e., $H_1 : \rho > 0$. Also, the empirical frequency of $I^* < -1.645$ corresponding to the rejection rates of the null hypothesis $H_0 : \rho = 0$ against the alternative hypothesis of negative spatial autocorrelation, i.e., $H_1 : \rho < 0$. Figure 1 shows the histograms of I^* for $n = 200$. These simulations confirm our theoretical results. In summary, the paper's main result is that, regardless of ρ, I^* asymptotically lies between $-\infty$ and $1/\sqrt{2}$ under the complete bipartite network. If ρ lies in a usual range satisfying spatial stability, $I^{*\prime}s$ asymptotic distribution might have an upper bound of $1/\sqrt{2}$. However, if ρ is close to -1, I^* would exhibit a different pattern, converging to $-\infty$. This can be verified by comparing the first panel of Figure 1 (odd pattern if $\rho = -0.99$) with the other panels of Figure 1 (highest density around $1/\sqrt{2}$).

Acknowledgements

We would like to thank an anonymous referee whose comments improved our paper.

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[2]Additional results of $\frac{p}{n} = 0.1$ and 0.5 are available in the supplemental Appendix available upon request from the authors.

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Table 1: Simulation Results of the Standardized Moran I Test Statistic I^* ($p/n = 0.3$)

$n = 50$

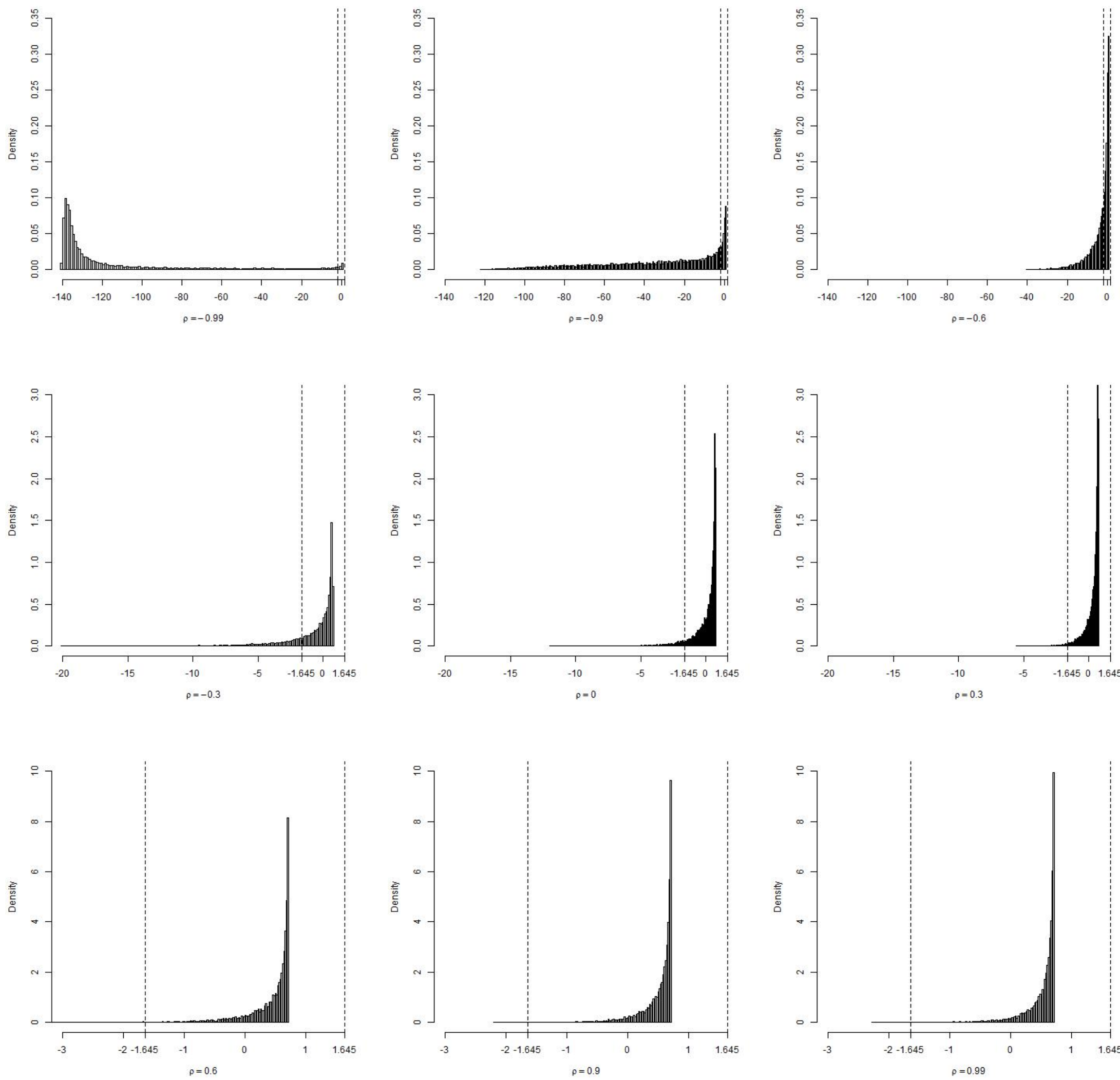
ρ	-0.99	-0.9	-0.6	-0.3	0	0.3	0.6	0.9	0.99
Min	-34.704	-33.625	-22.886	-15.506	-12.199	-5.228	-4.304	-3.039	-3.134
1st Quartile	-33.803	-24.381	-4.391	-1.192	-0.241	0.105	0.312	0.422	0.443
Median	-32.811	-15.795	-1.223	0.032	0.379	0.514	0.584	0.623	0.632
Mean	-30.638	-15.012	-2.720	-0.668	-0.005	0.266	0.414	0.497	0.509
3rd Quartile	-30.877	-5.098	0.275	0.573	0.652	0.682	0.697	0.706	0.706
Max	0.727	0.735	0.742	0.742	0.750	0.747	0.750	0.761	0.746
Frequency of $I^* > 1.645$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Frequency of $I^* < -1.645$	0.986	0.844	0.453	0.200	0.071	0.023	0.005	0.002	0.001

$n = 200$

ρ	-0.99	-0.9	-0.6	-0.3	0	0.3	0.6	0.9	0.99
Min	-140.522	-124.453	-50.342	-30.464	-9.048	-6.141	-3.994	-2.102	-1.846
1st Quartile	-137.328	-56.025	-4.752	-1.194	-0.238	0.138	0.331	0.451	0.464
Median	-133.235	-26.020	-1.269	0.047	0.385	0.523	0.580	0.621	0.629
Mean	-117.641	-33.845	-3.354	-0.718	0.005	0.282	0.422	0.514	0.528
3rd Quartile	-117.029	-6.364	0.257	0.566	0.641	0.671	0.682	0.692	0.694
Max	0.712	0.713	0.713	0.713	0.713	0.715	0.714	0.714	0.714
Frequency of $I^* > 1.645$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Frequency of $I^* < -1.645$	0.986	0.859	0.458	0.200	0.065	0.019	0.004	0.001	0.000

Note: 10,000 replications.

Figure 1: Histogram of the Standardized Moran I Test Statistic I^* ($n = 200, p/n = 0.3$)



Supplemental Appendix

This supplemental appendix provides proofs and extra Monte Carlo results which are not intended for publication due to space constraints.

A Additional Monte Carlo Simulation Results of the Moran Test

In this section, we present additional the Monte Carlo simulation results of the Moran test. Table 2 and Figure 2 report the results of $\frac{p}{n} = 0.1$. Table 3 and Figure 3 report the results of $\frac{p}{n} = 0.5$. Overall, their results are similar to those of $\frac{p}{n} = 0.3$ reported in the paper.

Table 2: Simulation Results of the Standardized Moran I Test Statistic I^* ($p/n = 0.1$)

$n = 50$

ρ	-0.99	-0.9	-0.6	-0.3	0	0.3	0.6	0.9	0.99
Min	-30.402	-29.558	-22.988	-14.751	-8.829	-4.491	-4.362	-2.545	-2.962
1st Quartile	-29.807	-22.942	-4.339	-1.142	-0.252	0.146	0.321	0.429	0.437
Median	-29.131	-15.692	-1.171	0.045	0.384	0.526	0.588	0.626	0.628
Mean	-27.291	-14.278	-2.668	-0.654	-0.004	0.291	0.421	0.500	0.513
3rd Quartile	-27.704	-5.260	0.268	0.573	0.652	0.683	0.697	0.704	0.705
Max	0.730	0.745	0.750	0.750	0.754	0.751	0.752	0.755	0.755
Frequency of $I^* > 1.645$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Frequency of $I^* < -1.645$	0.987	0.847	0.447	0.196	0.072	0.019	0.006	0.002	0.001

$n = 200$

ρ	-0.99	-0.9	-0.6	-0.3	0	0.3	0.6	0.9	0.99
Min	-136.177	-124.292	-50.456	-17.371	-11.732	-7.452	-3.250	-2.880	-2.614
1st Quartile	-133.245	-54.416	-5.099	-1.160	-0.208	0.148	0.341	0.444	0.459
Median	-129.508	-25.402	-1.291	0.064	0.388	0.512	0.587	0.621	0.627
Mean	-114.329	-33.056	-3.443	-0.676	0.013	0.287	0.433	0.506	0.524
3rd Quartile	-114.486	-6.160	0.240	0.574	0.643	0.667	0.684	0.692	0.693
Max	0.714	0.715	0.715	0.715	0.715	0.716	0.716	0.716	0.716
Frequency of $I^* > 1.645$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Frequency of $I^* < -1.645$	0.985	0.856	0.464	0.196	0.065	0.017	0.004	0.001	0.001

Note: 10,000 replications.

Figure 2: Histogram of the Standardized Moran I Test Statistic I^* $(n = 200, p/n = 0.1)$

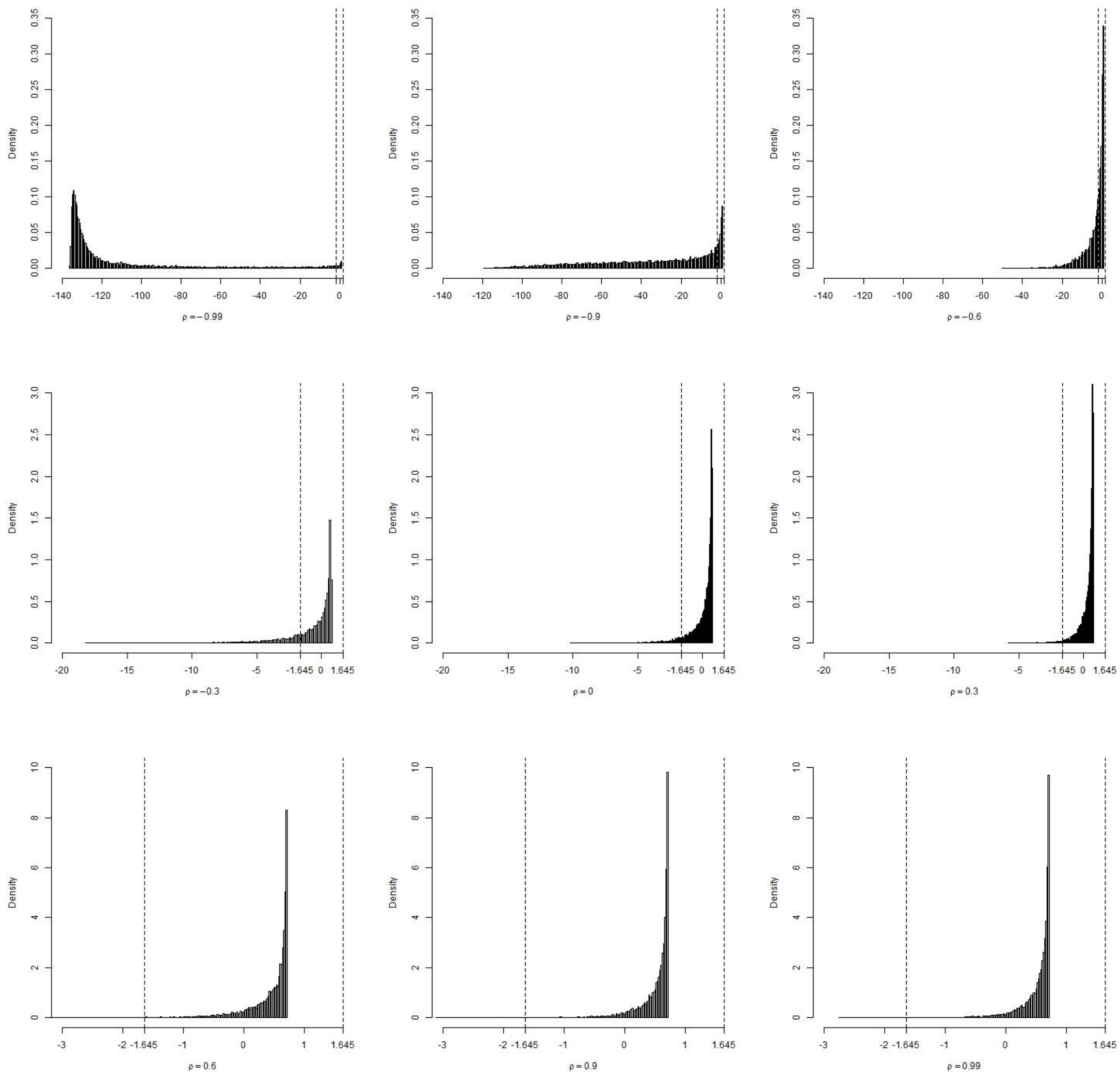


Table 3: Simulation Results of the Standardized Moran I Test Statistic I^* ($p/n = 0.5$)

$n = 50$

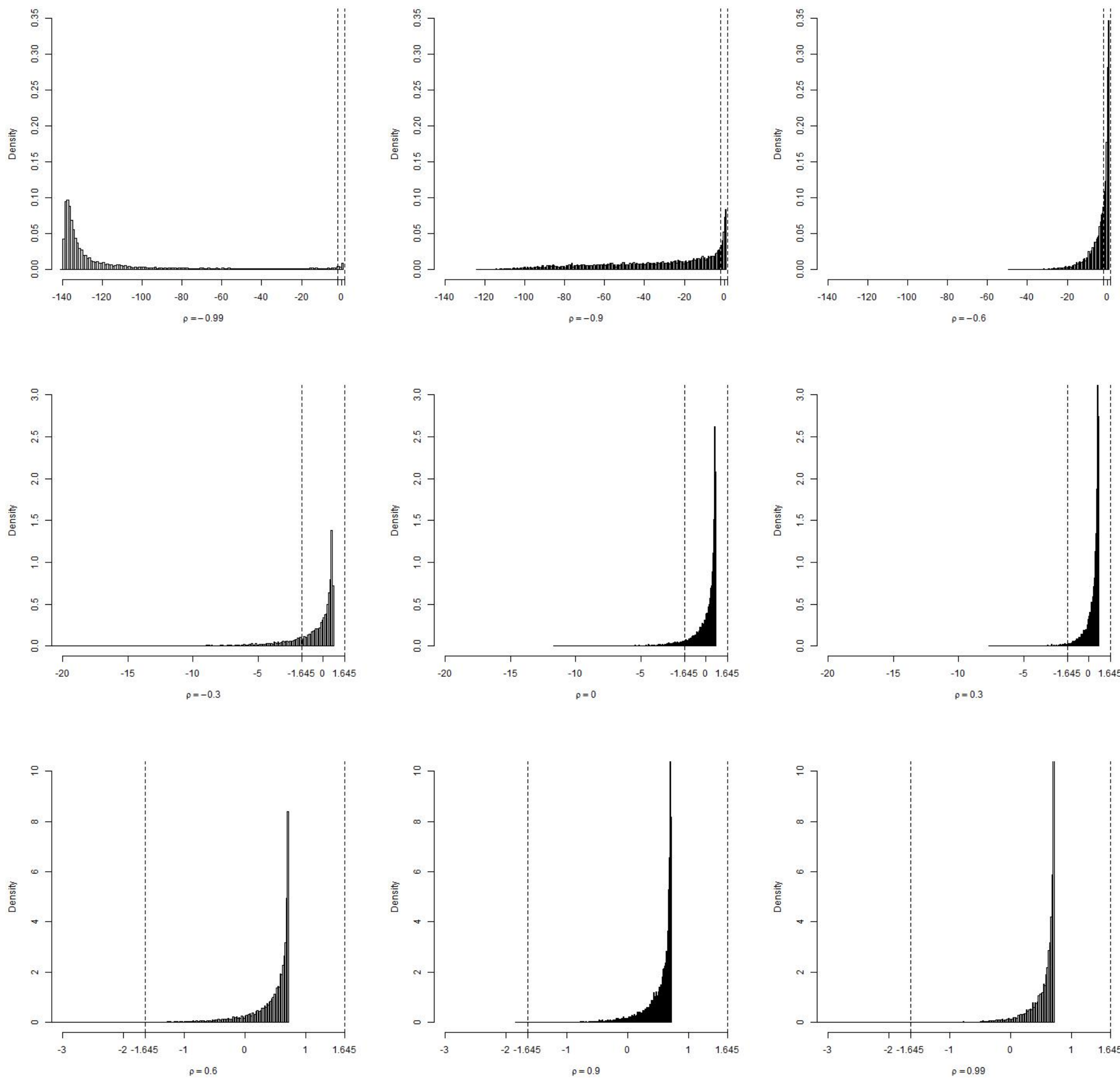
ρ	-0.99	-0.9	-0.6	-0.3	0	0.3	0.6	0.9	0.99
Min	-34.255	-33.305	-27.457	-16.688	-8.823	-5.505	-3.070	-2.464	-5.823
1st Quartile	-33.363	-24.338	-4.370	-1.211	-0.249	0.127	0.311	0.422	0.443
Median	-32.396	-16.097	-1.212	0.059	0.393	0.523	0.585	0.626	0.632
Mean	-30.168	-15.180	-2.699	-0.661	0.009	0.280	0.416	0.493	0.513
3rd Quartile	-30.428	-5.521	0.294	0.580	0.654	0.686	0.697	0.707	0.708
Max	0.738	0.747	0.751	0.757	0.758	0.767	0.764	0.756	0.757
Frequency of $I^* > 1.645$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Frequency of $I^* < -1.645$	0.984	0.854	0.452	0.201	0.068	0.022	0.006	0.002	0.001

$n = 200$

ρ	-0.99	-0.9	-0.6	-0.3	0	0.3	0.6	0.9	0.99
Min	-140.108	-122.600	-53.443	-21.542	-8.710	-5.486	-3.522	-2.912	-2.082
1st Quartile	-136.951	-56.609	-4.996	-1.207	-0.248	0.141	0.326	0.444	0.461
Median	-132.993	-26.710	-1.258	0.046	0.376	0.513	0.580	0.621	0.627
Mean	-117.116	-34.086	-3.391	-0.727	-0.007	0.281	0.423	0.508	0.525
3rd Quartile	-116.913	-6.113	0.259	0.566	0.638	0.668	0.682	0.692	0.693
Max	0.712	0.714	0.714	0.715	0.714	0.715	0.716	0.715	0.715
Frequency of $I^* > 1.645$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Frequency of $I^* < -1.645$	0.986	0.854	0.462	0.201	0.070	0.020	0.005	0.000	0.000

Note: 10,000 replications.

Figure 3: Histogram of the Standardized Moran I Test Statistic I^* $(n = 200, p/n = 0.5)$



B Proofs

Lemma 1 *Under Assumption 1, we have*

1.

$$tr\left(E_n W_n^2\right) = tr\left(E_n W_n^2 E_n W_n^2\right) = 1,$$

2.

$$\frac{1}{n}\tilde{X}_n' W_n^2 \tilde{X}_n = o_p\left(1\right)$$

3.

$$tr\left(M_n W_n^2\right) = 1 + o_p\left(1\right)$$

4.

$$tr\left(M_n W_n^2 M_n W_n^2\right) = 1 + o_p\left(1\right)$$

Proof. (1) Note that

$$W_n^2 \iota_n = \begin{bmatrix} \bar{J}_p & 0_{pq} \\ 0_{qp} & \bar{J}_q \end{bmatrix} \begin{bmatrix} \iota_p \\ \iota_q \end{bmatrix} = \begin{bmatrix} \iota_p \\ \iota_q \end{bmatrix} = \iota_n$$

Using $\bar{J}_n = \frac{1}{n}\iota_n \iota_n'$, we have

$$W_n^2 \bar{J}_n = \bar{J}_n$$

and hence

$$E_n W_n^2 = \left(I_n - \bar{J}_n\right) W_n^2 = W_n^2 - \bar{J}_n W_n^2 = W_n^2 - \bar{J}_n,$$

since E_n, W_n^2 and $\bar{J}_n$ are symmetric. Using the two equations above, we get

$$W_n^2 E_n W_n^2 = W_n^2\left(W_n^2 - \bar{J}_n\right) = W_n^2 - \bar{J}_n.$$

and hence

$$E_n W_n^2 E_n W_n^2 = E_n\left(W_n^2 - \bar{J}_n\right) = E_n W_n^2$$

since $E_n \bar{J}_n = 0$. Note that $tr\left(\bar{J}_n\right) = 1$ and $tr\left(W_n^2\right) = tr\left(\bar{J}_p\right) + tr\left(\bar{J}_q\right) = 2$ using $tr\left(\bar{J}_p\right) = 1$ and $tr\left(\bar{J}_q\right) = 1$. Therefore,

$$tr\left(E_n W_n^2 E_n W_n^2\right) = tr\left(E_n W_n^2\right) = tr\left(W_n^2\right) - tr\left(\bar{J}_n\right) = 1.$$

(2) We have

$$W_n^2 \tilde{X}_n = \begin{bmatrix} \bar{J}_p & 0_{pq} \\ 0_{qp} & \bar{J}_q \end{bmatrix} \begin{bmatrix} X_p - \iota_p \bar{X} \\ X_q - \iota_q \bar{X} \end{bmatrix} = \begin{bmatrix} \iota_p\left(\bar{X}_p - \bar{X}\right) \\ \iota_q\left(\bar{X}_q - \bar{X}\right) \end{bmatrix},$$

where $\bar{X}_p = \frac{1}{p}\iota_p' X_p$ and $\bar{X}_q = \frac{1}{q}\iota_q' X_q$. Because W_n^2 is symmetric and idempotent, i.e. $W_n^2 = W_n^2 W_n^2$, we have

$$\begin{aligned}
\tilde{X}_n' W_n^2 \tilde{X}_n &= \tilde{X}_n' W_n^2 W_n^2 \tilde{X}_n = \begin{bmatrix} \left(\bar{X}_p - \bar{X}\right)' \iota_p' & \left(\bar{X}_q - \bar{X}\right)' \iota_q' \end{bmatrix} \begin{bmatrix} \iota_p \left(\bar{X}_p - \bar{X}\right) \\ \iota_q \left(\bar{X}_q - \bar{X}\right) \end{bmatrix} \\
&= p \left(\bar{X}_p - \bar{X}\right)' \left(\bar{X}_p - \bar{X}\right) + q \left(\bar{X}_q - \bar{X}\right)' \left(\bar{X}_q - \bar{X}\right)
\end{aligned}$$

Since

$$\bar{X} = \frac{1}{n}\iota_n' X_n = \frac{1}{n}\begin{bmatrix} \iota_p' & \iota_q' \end{bmatrix} \begin{bmatrix} X_p \\ X_q \end{bmatrix} = \frac{1}{n}\left(\iota_p' X_p + \iota_q' X_q\right) = \frac{p}{n}\bar{X}_p + \frac{q}{n}\bar{X}_q,$$

we have

$$\bar{X}_p - \bar{X} = \bar{X}_p - \left(\frac{p}{n}\bar{X}_p + \frac{q}{n}\bar{X}_q\right) = \frac{q}{n}\left(\bar{X}_p - \bar{X}_q\right)$$

and similarly

$$\bar{X}_q - \bar{X} = \bar{X}_q - \left(\frac{p}{n}\bar{X}_p + \frac{q}{n}\bar{X}_q\right) = -\frac{p}{n}\left(\bar{X}_p - \bar{X}_q\right)$$

Hence

$$\begin{aligned}
\frac{1}{n}\tilde{X}_n' W_n^2 \tilde{X}_n &= \frac{p}{n}\left(\bar{X}_p - \bar{X}\right)' \left(\bar{X}_p - \bar{X}\right) + \frac{q}{n}\left(\bar{X}_q - \bar{X}\right)' \left(\bar{X}_q - \bar{X}\right) \\
&= \frac{p}{n}\frac{q^2}{n^2}\left(\bar{X}_p - \bar{X}_q\right)' \left(\bar{X}_p - \bar{X}_q\right) + \frac{q}{n}\frac{p^2}{n^2}\left(\bar{X}_p - \bar{X}_q\right)' \left(\bar{X}_p - \bar{X}_q\right) \\
&= \frac{pq}{n^2}\left(\bar{X}_p - \bar{X}_q\right)' \left(\bar{X}_p - \bar{X}_q\right) \\
&= o_p(1)
\end{aligned}$$

using Assumption 1.

(3) By Lemma 2 in Ding (2021), we have

$$P_n = \bar{J}_n + \tilde{X}_n \left(\tilde{X}_n' \tilde{X}_n\right)^{-1} \tilde{X}_n'$$

Hence

$$M_n = I_n - P_n = I_n - \left[\bar{J}_n + \tilde{X}_n \left(\tilde{X}_n' \tilde{X}_n\right)^{-1} \tilde{X}_n'\right] = E_n - \tilde{X}_n \left(\tilde{X}_n' \tilde{X}_n\right)^{-1} \tilde{X}_n'$$

and

$$M_n W_n^2 = \left[E_n - \tilde{X}_n \left(\tilde{X}_n' \tilde{X}_n\right)^{-1} \tilde{X}_n'\right] W_n^2 = E_n W_n^2 - \tilde{X}_n \left(\tilde{X}_n' \tilde{X}_n\right)^{-1} \tilde{X}_n' W_n^2.$$

As shown in Lemma 1.1, $tr\left(E_n W_n^2\right) = 1$. Using Assumption 1 and the result in Lemma 1.2, we get

$$tr\left[\tilde{X}_n \left(\tilde{X}_n' \tilde{X}_n\right)^{-1} \tilde{X}_n' W_n^2\right] = tr\left[\left(\tilde{X}_n' \tilde{X}_n\right)^{-1} \tilde{X}_n' W_n^2 \tilde{X}_n\right] = tr\left[\left(\frac{1}{n}\tilde{X}_n' \tilde{X}_n\right)^{-1} \left(\frac{1}{n}\tilde{X}_n' W_n^2 \tilde{X}_n\right)\right] = o_p(1).$$

Therefore, we obtain

$$tr\left(M_nW_n^2\right)=tr\left(E_nW_n^2\right)-tr\left[\tilde{X}_n\left(\tilde{X}_n'\tilde{X}_n\right)^{-1}\tilde{X}_n'W_n^2\right]=1+o_p\left(1\right)$$

(4) Also,

$$\begin{aligned}M_nW_n^2M_nW_n^2 &= \left[E_nW_n^2-\tilde{X}_n\left(\tilde{X}_n'\tilde{X}_n\right)^{-1}\tilde{X}_n'W_n^2\right]\left[E_nW_n^2-\tilde{X}_n\left(\tilde{X}_n'\tilde{X}_n\right)^{-1}\tilde{X}_n'W_n^2\right]\\ &= E_nW_n^2E_nW_n^2-\tilde{X}_n\left(\tilde{X}_n'\tilde{X}_n\right)^{-1}\tilde{X}_n'W_n^2E_nW_n^2\\ &\quad -E_nW_n^2\tilde{X}_n\left(\tilde{X}_n'\tilde{X}_n\right)^{-1}\tilde{X}_n'W_n^2+\tilde{X}_n\left(\tilde{X}_n'\tilde{X}_n\right)^{-1}\tilde{X}_n'W_n^2\tilde{X}_n\left(\tilde{X}_n'\tilde{X}_n\right)^{-1}\tilde{X}_n'W_n^2\end{aligned}$$

As shown in Lemma 1.1, $tr\left(E_nW_n^2E_nW_n^2\right)=1$. Using $W_n^2E_nW_n^2=E_nW_n^2$ in Lemma 1.1 and $E_n\tilde{X}_n=\tilde{X}_n$, we get

$$\begin{aligned}tr\left[\tilde{X}_n\left(\tilde{X}_n'\tilde{X}_n\right)^{-1}\tilde{X}_n'W_n^2E_nW_n^2\right] &= tr\left[\left(\frac{1}{n}\tilde{X}_n'\tilde{X}_n\right)^{-1}\left(\frac{1}{n}\tilde{X}_n'W_n^2\tilde{X}_n\right)\right]=o_p\left(1\right),\\ tr\left[E_nW_n^2\tilde{X}_n\left(\tilde{X}_n'\tilde{X}_n\right)^{-1}\tilde{X}_n'W_n^2\right] &= tr\left[\left(\frac{1}{n}\tilde{X}_n'\tilde{X}_n\right)^{-1}\left(\frac{1}{n}\tilde{X}_n'W_n^2\tilde{X}_n\right)\right]=o_p\left(1\right),\end{aligned}$$

and

$$\begin{aligned}&tr\left[\tilde{X}_n\left(\tilde{X}_n'\tilde{X}_n\right)^{-1}\tilde{X}_n'W_n^2\tilde{X}_n\left(\tilde{X}_n'\tilde{X}_n\right)^{-1}\tilde{X}_n'W_n^2\right]\\ =\ &tr\left[\left(\frac{1}{n}\tilde{X}_n'\tilde{X}_n\right)^{-1}\left(\frac{1}{n}\tilde{X}_n'W_n^2\tilde{X}_n\right)\left(\frac{1}{n}\tilde{X}_n'\tilde{X}_n\right)^{-1}\left(\frac{1}{n}\tilde{X}_n'W_n^2\tilde{X}_n\right)\right]=o_p\left(1\right).\end{aligned}$$

Therefore,

$$tr\left(M_nW_n^2M_nW_n^2\right)=1+o_p\left(1\right)$$

■

Lemma 2 *Under Assumption 1, we have*

1.

$$\left(n-k\right)\mu_I=-1+o_p\left(1\right),$$

2.

$$\left(n-k\right)\sigma_I=\sqrt{2}+o_p\left(1\right).$$

Proof. (1) $\left(n-k\right)\mu_I=tr\left(M_nW_n\right)=-tr\left(M_nW_n^2\right)=-1+o_p\left(1\right)$ using Lemma 1.

(2) Since $M_nW_n = -M_nW_n^2$ and W_n^2 is symmetric, we have

$$tr\left(M_nW_nM_nW_n\right) = tr\left(M_nW_n^2M_nW_n^2\right)$$

In addition, since $M_n = M_nM_n$ and W_n^2 is symmetric, we have

$$\begin{aligned} tr\left(M_nW_nM_nW_n'\right) &= tr\left(M_nW_nM_nW_n'M_n\right) = tr\left[\left(M_nW_n\right)M_n\left(M_nW_n\right)'\right] \\ &= tr\left[\left(M_nW_n^2\right)M_n\left(M_nW_n^2\right)'\right] = tr\left[M_nW_n^2M_nW_n^2M_n\right] = tr\left(M_nW_n^2M_nW_n^2\right). \end{aligned}$$

Using Lemma 1, we get

$$\begin{aligned} (n-k)\,\sigma_I &= (n-k)\sqrt{\frac{tr\left(M_nW_nM_nW_n'\right) + tr\left(M_nW_nM_nW_n\right) - \frac{2}{n-k}\left[tr\left(M_nW_n\right)\right]^2}{(n-k)\,(n-k+2)}} \\ &= \sqrt{\frac{n-k}{n-k+2}\left\{2tr\left(M_nW_n^2M_nW_n^2\right) - \frac{2}{n-k}\left[tr\left(M_nW_n^2\right)\right]^2\right\}} \\ &= \sqrt{2} + o_p(1). \end{aligned}$$

■

B.1 Proof of Theorem 1

Proof. Because

$$I^* = \frac{I-\mu_I}{\sigma_I} \leq -\frac{\mu_I}{\sigma_I}.$$

where

$$-\frac{\mu_I}{\sigma_I} = \frac{-(n-k)\,\mu_I}{(n-k)\,\sigma_I} = \frac{1}{\sqrt{2}} + o_p(1)$$

using Lemma 2. This implies that

$$\lim_{n\to\infty} \Pr\left(I^* > \frac{1}{\sqrt{2}}\right) = 0$$

for all ρ. ■

Lemma 3 *1.*

$$M_nB_n^{-1}\left(B_n^{-1}\right)'M_n = M_n - \frac{\rho(2+\rho)}{(1+\rho)^2}M_nW_n^2M_n,$$

2.

$$M_nB_n^{-1}\left(B_n^{-1}\right)'M_nW_n = M_nW_n^2 + \frac{\rho(2+\rho)}{(1+\rho)^2}M_nW_n^2M_nW_n^2,$$

Proof. (1) Using the result $M_n W_n^2 = -M_n W_n$ in Section 2, we have

$$\begin{aligned}
M_n W_n^3 &= \left(M_n W_n^2\right) W_n = \left(-M_n W_n\right) W_n = -M_n W_n^2 = M_n W_n, \\
M_n W_n^4 &= \left(M_n W_n^3\right) W_n = \left(M_n W_n\right) W_n = M_n W_n^2 = -M_n W_n, \\
&\vdots
\end{aligned}$$

Hence

$$\begin{aligned}
M_n B_n^{-1} &= M_n \left(I_n - \rho W_n\right)^{-1} \\
&= M_n + \rho M_n W_n + \rho^2 M_n W_n^2 + \rho^3 M_n W_n^3 + \cdots \\
&= M_n + \left(\rho - \rho^2 + \rho^3 - \cdots\right) M_n W_n \\
&= M_n + \frac{\rho}{1+\rho} M_n W_n \\
&= M_n - \frac{\rho}{1+\rho} M_n W_n^2.
\end{aligned}$$

Note that W_n^2 and M_n are symmetric and idempotent, with

$$\begin{aligned}
M_n B_n^{-1} \left(B_n^{-1}\right)' M_n &= M_n B_n^{-1} \left(M_n B_n^{-1}\right)' \\
&= \left(M_n - \frac{\rho}{1+\rho} M_n W_n^2\right) \left(M_n - \frac{\rho}{1+\rho} M_n W_n^2\right)' \\
&= \left(M_n - \frac{\rho}{1+\rho} M_n W_n^2\right) \left(M_n - \frac{\rho}{1+\rho} W_n^2 M_n\right) \\
&= M_n - \frac{2\rho}{1+\rho} M_n W_n^2 M_n + \frac{\rho^2}{\left(1+\rho\right)^2} M_n W_n^2 M_n \\
&= M_n - \frac{\rho\left(2+\rho\right)}{\left(1+\rho\right)^2} M_n W_n^2 M_n.
\end{aligned}$$

(2) Using the fact that M_n is idempotent, we further get

$$\begin{aligned}
M_n B_n^{-1} \left(B_n^{-1}\right)' M_n W_n &= \left[M_n B_n^{-1} \left(B_n^{-1}\right)' M_n\right] M_n W_n \\
&= -\left[M_n - \frac{\rho\left(2+\rho\right)}{\left(1+\rho\right)^2} M_n W_n^2 M_n\right] M_n W_n^2 \\
&= -M_n W_n^2 + \frac{\rho\left(2+\rho\right)}{\left(1+\rho\right)^2} M_n W_n^2 M_n W_n^2.
\end{aligned}$$

■

Lemma 4 *Under Assumption 1, we have*

1.

$$\frac{1}{n-k} \hat{u}_n' \hat{u}_n = \sigma^2 + o_p\left(1\right),$$

2.
$$\hat{u}_n' W_n \hat{u}_n - \mu_I \hat{u}_n' \hat{u}_n = \frac{\rho(2+\rho)}{(1+\rho)^2}\sigma^2 + o_p(1).$$

Proof. (1) Since $\hat{u}_n = M_n u_n = M_n B_n^{-1}\varepsilon_n$, we get

$$\hat{u}_n' \hat{u}_n = \varepsilon_n' \left(B_n^{-1}\right)' M_n B_n^{-1}\varepsilon_n$$

Hence

$$\begin{aligned} E\left(\hat{u}_n' \hat{u}_n\right) &= tr\left[\left(B_n^{-1}\right)' M_n B_n^{-1}\right]\sigma^2 \\ &= tr\left[M_n B_n^{-1}\left(B_n^{-1}\right)' M_n\right]\sigma^2 \\ &= \left[tr\left(M_n\right) - \frac{\rho(2+\rho)}{(1+\rho)^2} tr\left(M_n W_n^2 M_n\right)\right]\sigma^2 \\ &= \left[n - k - \frac{\rho(2+\rho)}{(1+\rho)^2} tr\left(M_n W_n^2\right)\right]\sigma^2 \end{aligned}$$

using Lemma 3 and $tr(M_n) = n - k$. Therefore

$$\hat{u}_n' \hat{u}_n = \left[n - k - \frac{\rho(2+\rho)}{(1+\rho)^2}\right]\sigma^2 + o_p(1)$$

using Lemma 1.

(2) Similarly,

$$\hat{u}_n' W_n \hat{u}_n = \varepsilon_n' \left(B_n^{-1}\right)' M_n W_n M_n B_n^{-1}\varepsilon_n$$

Hence

$$\begin{aligned} E\left(\hat{u}_n' W_n \hat{u}_n\right) &= tr\left[\left(B_n^{-1}\right)' M_n W_n M_n B_n^{-1}\right]\sigma^2 \\ &= tr\left[M_n W_n M_n B_n^{-1}\left(B_n^{-1}\right)' M_n\right]\sigma^2 \\ &= -tr\left(M_n W_n^2\right)\sigma^2 + \frac{\rho(2+\rho)}{(1+\rho)^2} tr\left(M_n W_n^2 M_n W_n^2\right)\sigma^2 \end{aligned}$$

using Lemma 3. Also

$$\hat{u}_n' W_n \hat{u}_n = \left[-1 + \frac{\rho(2+\rho)}{(1+\rho)^2}\right]\sigma^2 + o_p(1)$$

using Lemma 1.

Therefore,

$$\begin{aligned}
& \hat{u}_n' W_n \hat{u}_n - \mu_I \hat{u}_n' \hat{u}_n \\
= & \hat{u}_n' W_n \hat{u}_n + \frac{1}{n-k} \hat{u}_n' \hat{u}_n - \left[(n-k)\, \mu_I + 1 \right] \frac{1}{n-k} \hat{u}_n' \hat{u}_n \\
= & \left[-1 + \frac{\rho\,(2+\rho)}{(1+\rho)^2} \right] \sigma^2 - \frac{1}{n-k} \left[n - k - \frac{\rho\,(2+\rho)}{(1+\rho)^2} \right] \sigma^2 + o_p(1) \\
= & \frac{n-k+1}{n-k} \frac{\rho\,(2+\rho)}{(1+\rho)^2} \sigma^2 + o_p(1) .
\end{aligned}$$

■

C Proof of Theorem 2

Proof. When $\rho = -1 + \frac{1}{\psi_n}$, we have

$$\frac{\rho\,(2+\rho)}{(1+\rho)^2} = \frac{\left(-1 + \frac{1}{\psi_n}\right)\left(1 + \frac{1}{\psi_n}\right)}{\left(\frac{1}{\psi_n}\right)^2} = 1 - \psi_n^2$$

Hence

$$\begin{aligned}
\hat{u}_n' \hat{u}_n &= \left[n - k - \frac{\rho\,(2+\rho)}{(1+\rho)^2} \right] \sigma^2 + o_p(1) \\
&= \left(n - k - 1 + \psi_n^2 \right) \sigma^2 + o_p(1)
\end{aligned}$$

and

$$\begin{aligned}
\hat{u}_n' W_n \hat{u}_n - \mu_I \hat{u}_n' \hat{u}_n &= \frac{n-k+1}{n-k} \frac{\rho\,(2+\rho)}{(1+\rho)^2} \sigma^2 + o_p(1) \\
&= \frac{n-k+1}{n-k} \left(1 - \psi_n^2\right) \sigma^2 + o_p(1)
\end{aligned}$$

so that

$$I - \mu_I = \frac{\hat{u}_n' W_n \hat{u}_n}{\hat{u}_n' \hat{u}_n} - \mu_I = \frac{\hat{u}_n' W_n \hat{u}_n - \mu_I \hat{u}_n' \hat{u}_n}{\hat{u}_n' \hat{u}_n} = \frac{\frac{n-k+1}{n-k}\left(1 - \psi_n^2\right) \sigma^2}{\left(n - k - 1 + \psi_n^2\right) \sigma^2} + o_p(1) = \frac{(n-k+1)\left(1 - \psi_n^2\right)}{(n-k)\left(n - k - 1 + \psi_n^2\right)} + o_p(1)$$

using Lemma 2. Therefore

$$I^* = \frac{(n-k)\,(I - \mu_I)}{(n-k)\,\sigma_I} = \frac{(n-k+1)\left(1 - \psi_n^2\right)}{\sqrt{2}\left(n - k - 1 + \psi_n^2\right)} + o_p(1) = \begin{cases} O_p\left(-\psi_n^2\right), & \text{if } \psi_n^2 << n \\ O_p(-n), & \text{if } \psi_n^2 >> n \end{cases}$$

using Lemma 1, which implies $I^* \xrightarrow{p} -\infty$ if $\psi_n \to \infty$ as $n \to \infty$. This means that

$$\lim_{n \to \infty} \Pr\left(I^* < \eta\right) = 1$$

for every constant η. ■

C.1 The LM test in a Spatial Lag Model under a Complete Bipartite Network

The following spatial lag model was considered by Baltagi and Liu (2009):

$$y_n = \rho W_n y_n + X_n \beta + \varepsilon_n \tag{C1}$$

where y_n is an $n \times 1$ vector for the dependent variable. ι_n is a vector of ones of dimension n. X_n is an $n \times k$ matrix of exogenous variables including a constant. β is a $k \times 1$ vector of parameters. ρ is a scalar parameter between -1 and 1. u_n and ε_n are $n \times 1$ vectors, where ε_n is independent and identically distributed as Normal with zero mean and variance σ^2. The LM test statistic for the null hypothesis of $H_0 : \rho = 0$ is given by:

$$LM = \frac{\left(\hat{u}_n' W_n y_n / \hat{\sigma}^2\right)^2}{\tilde{D}_n + T_n}$$

where $\hat{u}_n$ is the OLS residual from regressing y_n on X_n, and $\hat{\sigma}^2 = \frac{1}{n}\hat{u}_n'\hat{u}_n$. $\tilde{D}_n = \left(W_n X_n \hat{\beta}\right)' M_n W_n X_n \hat{\beta} / \hat{\sigma}^2$ where $\hat{\beta}$ is the OLS estimator, $T_n = tr\left(W_n^2 + W_n' W_n\right)$. Baltagi and Liu (2009) showed that when $W_n = \frac{1}{(n-1)}(\iota_n \iota_n' - I_n)$, $LM \xrightarrow{p} \frac{1}{2}$.

In this Appendix, we check the performance of this LM test under a complete bipartite network spatial weighting matrix using simulations. We generate the data from Equation (C1), where $X_n = (\iota_n, x_{1n}, x_{2n})$ and $\beta = (5, 1, 1)'$. ι_n is a vector of ones of dimension n. x_{1n}, x_{2n} and ε_n are $n \times 1$ vectors with elements $x_{1i} \overset{iid}{\sim} \sqrt{6} U(0,1)$, $x_{2i} \overset{iid}{\sim} N(0,1)/\sqrt{2}$ and $\varepsilon_i \overset{iid}{\sim} N(0,1)$. ρ varies over the range $(-0.99, -0.9, -0.6, -0.3, 0, 0.3, 0.6, 0.9, 0.99)$. The $n \times n$ spatial weight matrix W_n is the complete bipartite network spatial weighting matrix. We let $\frac{p}{n} = 0.3$. The sample sizes considered are $n = (50, 200)$. For each experiment, we perform $10,000$ replications.

Table 4 reports the summary statistics for the LM test and the empirical frequency of $LM > 3.841$ corresponding to the rejection rates of the null hypothesis $H_0 : \rho = 0$ against the alternative hypothesis of no spatial autocorrelation $H_1 : \rho \neq 0$.[3] These simulations show that the LM test for spatial lag under a complete bipartite network spatial weighting matrix yield similar performance to that of the standardized Moran I^* test for the spatial error model. In particular, this LM test can never reject the null hypothesis of zero spatial correlation when the true ρ is positive. In contrast, this LM test will always reject the null hypothesis of zero spatial correlation when the true ρ is negative and close to -1.

[3]For a Chi-squared distribution with one degree of freedom, the critical value at the 0.05 significance level is 3.841.

Table 4: Simulation Results of the LM Test Statistic ($p/n = 0.3$)

$n = 50$

ρ	-0.99	-0.9	-0.6	-0.3	0	0.3	0.6	0.9	0.99
Min	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1st Quartile	470.711	38.480	0.170	0.014	0.004	0.002	0.001	0.001	0.001
Median	507.278	206.388	3.523	0.260	0.045	0.019	0.012	0.007	0.007
Mean	468.893	207.992	19.571	2.937	0.677	0.220	0.096	0.052	0.048
3rd Quartile	520.001	357.808	22.569	2.324	0.378	0.116	0.061	0.040	0.037
Max	530.469	520.195	308.540	132.715	55.266	16.736	7.560	3.583	3.971
Frequency of $LM > 3.841$	0.989	0.874	0.491	0.186	0.045	0.008	0.001	0.000	0.000

$n = 200$

ρ	-0.99	-0.9	-0.6	-0.3	0	0.3	0.6	0.9	0.99
Min	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1st Quartile	6879.291	53.859	0.168	0.012	0.004	0.002	0.001	0.001	0.001
Median	7945.421	732.233	3.637	0.218	0.044	0.018	0.010	0.007	0.006
Mean	6894.053	1423.008	35.373	3.523	0.695	0.199	0.084	0.047	0.041
3rd Quartile	8225.423	2454.621	28.683	2.170	0.378	0.113	0.054	0.038	0.032
Max	8404.211	7148.934	1156.906	372.931	38.683	14.767	5.070	3.370	2.288
Frequency of $LM > 3.841$	0.986	0.875	0.493	0.187	0.045	0.006	0.001	0.000	0.000

Note: 10,000 replications.

References

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